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# AN INFINITE SET OF EXOTIC $\mathbf{R}^4$'S

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## Introduction

In 1982, Michael Freedman startled the topological community by pointing out the existence of an "exotic $\mathbf{R}^4$", a smooth manifold homeomorphic to $\mathbf{R}^4$, but not diffeomorphic to it. This result follows easily from Donaldson's Theorem [2] on the nonexistence of certain smooth 4-manifolds, together with Freedman's powerful techniques [3] for analyzing 4-manifolds in the topological category. This exotic $\mathbf{R}^4$ was shocking to topologists, because in dimensions $n \neq 4$, it is a fundamental result of smoothing theory that there are no exotic $\mathbf{R}^n$'s. (Since $\mathbf{R}^n$ is contractible, there is no place for any bundle-theoretic obstruction to live.) Thus, this exotic $\mathbf{R}^4$ implies a catastrophic failure in dimension 4 of the basic philosophy of smoothing theory, as well as other high-dimensional techniques.

Freedman's discovery naturally raised questions about the set $\mathscr{R}$ of all oriented diffeomorphism types homeomorphic to $\mathbf{R}^4$. The most basic problem has been to determine the cardinality of $\mathscr{R}$. Soon after Freedman's result, the author showed [5] that $\mathscr{R}$ has at least four elements. More recently, Freedman and Taylor [4] have found a fifth element, a "universal" $\mathbf{R}^4$ in which all others must embed. In the present paper, we exploit a technique of Freedman and Taylor to prove that $\mathscr{R}$ is (at least countably) infinite.

Our main result, Theorem 2.3, asserts the existence of a doubly indexed family $\{R_{m,n} | m, n = 0, 1, 2, \cdots, \infty\}$ in $\mathscr{R}$ such that $R_{m,n}$ has an orientation-preserving embedding in $R_{m',n'}$ if and only if $m \leqslant m'$ and $n \leqslant n'$. In particular no two members of this family are related by an orientation-preserving diffeomorphism. We actually show that when $m > m'$ or $n > n'$ there is a compact subset of $R_{m,n}$ which cannot embed in $R_{m',n'}$.

We use these compact subsets to prove the required nonembedding property which distinguishes the $R_{m,n}$'s. Our key Lemma 1.2 gives a method for finding such compact sets which do not embed in a preassigned $R \in \mathscr{R}$. This is where

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we apply the method of Freedman and Taylor. Ultimately, the lemma is a consequence of Donaldson's Theorem (hence, gauge theory), specifically, this is needed to prove the existence of one compact set $X$ with a certain nonembedding property. The existence of $X$ follows from results in [5] which depend on Donaldson's Theorem.

In §1, we give our basic construction of a singly indexed family $\{R_n | n = 0, 1, 2, \cdots, \infty\}$. This is defined in Theorem 1.3. Its embedding properties are analogous to those of the doubly-indexed family. In §2, we sharpen the two main lemmas of the first section, so that each $R_n$ ($n$ finite) has an embedding in a sum of $\mathbf{C}P^2$'s. This allows us to construct the doubly indexed family of Theorem 2.3, with $R_{n,0} = R_n$. We conclude with an appendix. It shows that our basic method of gluing together $\mathbf{R}^4$'s defines a commutative monoid structure on $\mathscr{R}$, and gives some basic properties of $\mathscr{R}$ with this structure.

*Added in proof.* Clifford Taubes [10] has now proven a generalization of Donaldson's Theorem for open 4-manifolds with "periodic" ends. By a previous observation of Freedman, this implies the existence of an uncountable family of distinct $\mathbf{R}^4$'s, parametrized by $\mathbf{R}$. §3 describes this construction and enlarges the parameter space to $\mathbf{R}^2$. This section is independent of §§1 and 2, except for the introductory material preceding the lemmas of §1.

**Notation and conventions.** $\mathbf{R}^4$ will denote euclidean 4-space with the standard smooth structure. The letter $R$ will be used for oriented smooth manifolds homeomorphic to $\mathbf{R}^4$, or for the corresponding diffeomorphisms types in $\mathscr{R}$. (We will be careless with this distinction.) We may also refer to elements of $\mathscr{R}$ as "$\mathbf{R}^4$'s" (e.g. "exotic $\mathbf{R}^4$'s") when the meaning is clear from the context.

We will work in the oriented smooth category except when otherwise stated. Thus, maps will implicitly be smooth, and codimension zero embeddings (smooth or not) will preserve orientations. When a map or submanifold is not necessarily smooth, we will refer to it as a "topological" map or submanifold. (Homeomorphisms and flat embeddings are implicitly topological.) The notation $\overline{M}$ will denote the manifold $M$ with reversed orientation.

Disks and balls ($D^2, B^4$, etc.) will always be compact, unless otherwise stated.

Given a link $L$ in $S^3 = \partial B^4$, $M_L$ will denote the compact 4-manifold (with boundary) obtained by attaching 2-handles to a 4-ball along the link $L$, with framing zero. A smooth link $L$ with $k$ components is called *topologically* (or *smoothly*) *slice* if there is a topological (or smooth) embedding of $k$ disjoint copies of $(D^2 \times \mathbf{R}^2, \partial D^2 \times \mathbf{R}^2)$ into $(B^4, \partial B^4)$ such that the $k$ circles $\partial D^2 \times 0$ are mapped onto $L$. (We define $L$ to be slice in the connected sum $B^4 \# M$ if the above definition holds with $B^4$ replaced by $B^4 \# M$.) Notice that $L$ is slice in $B^4$ if and only if $M_L$ embeds in $S^4$ (or $\mathbf{R}^4$). (This holds in either the topological or smooth category.) To see this, assume $L$ is slice, then glue a 4-ball onto $B^4$ to obtain $S^4$. The new 4-ball union the slice disks will give $M_L$ embedded in $S^4$ (with reversed orientation).

We will make free use of Casson handles when necessary. These are smooth manifolds homeomorphic to $D^2 \times \mathbf{R}^2$, which are nested unions of certain compact subsets, called *towers*. Some references for this theory are [1], [3] and [7].

## 1. An infinite set of exotic $\mathbf{R}^4$'s

We would like to be able to glue together exotic $\mathbf{R}^4$'s in a manner analogous to that of boundary-sum of compact manifolds with boundary.

**Definition.** For $R, R' \in \mathscr{R}$ the *end-sum* $R \natural R'$ is defined as follows: Let $\gamma$: $[0, \infty) \to R$ and $\gamma'$: $[0, \infty) \to R'$ be smooth properly embedded rays with tubular neighborhoods $\nu$ and $\nu'$, respectively. Let $R \natural R' = R \cup_\phi I \times \mathbf{R}^3 \cup_{\phi'} R'$, where $\phi$: $[0, \frac{1}{2}) \times \mathbf{R}^3 \to \nu$ and $\phi'$: $(\frac{1}{2}, 1] \times \mathbf{R}^3 \to \nu'$ are orientation-preserving diffeomorphisms which respect the $\mathbf{R}^3$-bundle structures.

It is shown in the appendix that end-sum is well defined, and that it induces a commutative monoid structure on $\mathscr{R}$. We may avoid using these facts, however, by being careful when we construct end-sums.

In [5], an exotic $\mathbf{R}^4$ is constructed. It is denoted by $\mathbf{R}^4_\Gamma$ or $R_\Gamma$. This $R_\Gamma$ is constructed as a subset of $\mathbf{C}P^2$, and it has the property that it cannot be embedded in any smooth, closed, simply connected 4-manifold with a negative definite intersection form. The nonembedding property is proven by studying a certain topological collar $U$ (homeomorphic to $S^3 \times \mathbf{R}$) of the end of $R_\Gamma$. If $R$ is a neighborhood of $R_\Gamma - U$ with $R$ homeomorphic to $\mathbf{R}^4$, then $R \cap U$ will be a neighborhood of the end of $R$. Using this neighborhood, we may apply the same reasoning as for $R_\Gamma$ to show that $R$ has the same nonembedding property as $R_\Gamma$. We may take $R$ to have compact closure, so it is embedded in a smooth compact 4-manifold $X \subset R_\Gamma$ with $\partial X$ connected. This $X$ does not embed smoothly in any negative definite, closed, simply connected 4-manifold. We fix $X$ for the duration of the paper.

**Remark.** $X$ may be taken to be a fairly simple manifold. In particular, if $L$ is the fourth Whitehead double (with (–) clasps) of the left-hand trefoil knot, we may set $X = M_L$. This embeds in $R_\Gamma$ (as may be seen from the construction of $R_\Gamma$), and it has the required nonembedding property (by the methods of [6]).

We would like to improve our control over the end of $R_\Gamma$. To do this, we will shave a small collar off of the end of $R_\Gamma$, as described below.

**Definition.** $R \in \mathscr{R}$ is a *shaved* $\mathbf{R}^4$ if for some (possibly noncompact) smooth 4-manifold $M$ there is an embedding $i$: $R \hookrightarrow M$ with $i(R) = \text{int}\, B$, $B$ a flat topological 4-ball such that some neighborhood $U \subset \partial B$ is a smooth submanifold of $M$. The embedding $i$ will be called a *shaved embedding*.

In particular, we shave $R_\Gamma$ as follows: Fix a homeomorphism $h$: $\mathbf{R}^4 \to R_\Gamma$. Let $B \subset \mathbf{R}^4$ be a smooth ball of sufficiently large radius that $X \subset \text{int}\, h(B)$. Choose a point $x \in \partial B$. By the Stable Homeomorphism Theorem (Annulus Conjecture) of Quinn [8] we may perturb $h$ slightly near $x$ to obtain $h'$ which is smooth on some neighborhood of $x$. Define $R_1$ to be int $h'(B)$. Clearly, $R_1$ is a shaved $\mathbf{R}^4$, and the inclusion $R_1 \subset R_\Gamma$ is a shaved embedding. Furthermore, we have $X \subset R_1 \subset \mathbf{C}P^2$ with the last inclusion a shaved embedding. This $R_1$ will be the first element of our infinite sequence of exotic $\mathbf{R}^4$'s.

Notice that the end-sum of two shaved $\mathbf{R}^4$'s is shaved. In particular, suppose $R \hookrightarrow B \subset M$ and $R' \hookrightarrow B' \subset M'$ are shaved embeddings. We may assume $\partial M = \varnothing = \partial M'$. Form the connected sum $M \# M'$ away from $B$ and $B'$. Now find a smooth arc $\gamma$ in $M \# M'$ which runs from the smooth part of $\partial B$ to the smooth part of $\partial B'$ and hits $B \cup B'$ only at the endpoints. The set $B \cup B'$ union a closed tubular neighborhood of $\gamma$ is a flat 4-ball with interior diffeomorphic to $R \natural R'$, so we have constructed a shave embedding $R \natural R' \hookrightarrow M \# M'$.

We now introduce two lemmas, from which we construct an infinite family of distinct exotic $\mathbf{R}^4$'s.

**Lemma 1.1.** *For any topologically slice link $L$ there is a shaved* $\mathbf{R}^4$, *denoted $R_L$, such that $M_L$ embeds smoothly in $R_L$.*

*Proof.* Since $L$ is topologically slice, there is a topological (flat) embedding $i$: $M_L \hookrightarrow \mathbf{R}^4$. We will define a new smoth structure on $\mathbf{R}^4$ so that $i$ becomes smooth. First we define our atlas near $i(M_L)$ by declaring $i$ to be a diffeomorphic embedding. The complement, $\mathbf{R}^4 - \text{int}\, i(M_L)$, is a noncompact connected 4-manifold. Quinn [8] has shown that any such manifold may be smoothed. (The groups $\pi_i(\text{TOP}_4/\text{O}_4)$ vanish for $i = 0, 1, 2$, so there are no obstructions.) Since smoothings on 3-manifolds are essentially unique, we may fit the two pieces together to obtain a smoothing on all of $\mathbf{R}^4$. We realize $R_L$ by shaving this exotic $\mathbf{R}^4$, just as we did to obtain $R_1$ from $R_\Gamma$.

**Lemma 1.2.** *Let $R$ be a shaved* $\mathbf{R}^4$. *Then there is a topologically slice link $L$ for which the disjoint union $X \amalg M_L$ cannot be smoothly embedded in $R$.*

**Theorem 1.3.** *There is a family $\{R_n | n = 0, 1, 2, \cdots, \infty\}$ of* $\mathbf{R}^4$*'s, shaved (except for $R_\infty$), such that $R_m$ embeds in $R_n$ if and only if $m \leqslant n$.*

**Proof of Theorem 1.3.** We will define the family recursively, assuming Lemma 1.2. Let $R_0 = \mathbf{R}^4$. $R_1$ has already been defined as a shaved $\mathbf{R}^4$ containing $X$. (Thus, it cannot embed in $R_0$.) Now for $n > 1$, assume

$R_0, \cdots, R_{n-1}$ have been defined and satisfy the conditions of the theorem. Since $R_{n-1}$ is shaved, we may apply Lemma 1.2 to obtain a link $L$ for which $X \amalg M_L$ does not embed in $R_{n-1}$. Let $R_n = R_{n-1} \natural R_L$, $R_L$ as in Lemma 1.1. Then $R_n$ is a shaved $\mathbf{R}^4$. By construction, $X \amalg M_L$ embeds in $R_n$ (since $X \subset R_1 \subset R_{n-1}$). Thus, $R_n$ cannot embed in $R_{n-1}$ (or in $R_m$, $m < n$, since these embed in $R_{n-1}$).

Now let $R_\infty = \bigcup_{n=0}^{\infty} R_n$. Note that $R_\infty$ cannot embed in $R_n$, $n < \infty$, since it contains $R_{n+1}$.

**Remarks.** By construction, each embedding $R_m \subset R_n$ is as an end-summand. In the cases where no embedding $R_m \hookrightarrow R_n$ exists, there is a compact 4-manifold in $R_m$ which does not embed in $R_n$.

We may arrange $R_\infty$ to be the universal $\mathbf{R}^4$ of Freedman and Taylor [4] by choosing the links of Lemma 1.2 to be sufficiently complicated. In §2, we will prevent this by keeping the links within a certain subclass.

**Proof of Lemma 1.2.** Our method is essentially due to Freedman and Taylor [4], who used it to prove that their universal $\mathbf{R}^4$ does not embed in a flat ball in any 4-manifold.

We are given $R \subset B \subset M$; $B$ is a flat topological 4-ball in some smooth 4-manifold $M$. (This is as much as we need of the hypothesis that $R$ is shaved.) We show that $M$ may be taken to be closed and simply connected, with hyperbolic intersection form. (We may even assume that $M$ is diffeomorphic to a connected sum of $S^2 \times S^2$'s.) To see this, first note that since $B$ is flat, it is surrounded by a topological collar homeomorphic to $S^3 \times \mathbf{R}$ in $M - B$. This collar contains a smooth 3-manifold surrounding $B$. (For example, perturb the projection $S^3 \times \mathbf{R} \to \mathbf{R}$ to get a smooth proper map, then pull back a regular value.) Throw away everything outside of this 3-manifold to obtain $M'$, a compact smooth 4-manifold with boundary, such that $B$ is flat in int $M'$. Note that $M'$ is spin, since it is embedded in the contractible manifold $B \cup$ (collar). Now take the double $2M' = \partial(M' \times I)$. This is closed, smooth (after rounding the edges) and spin. Furthermore, the signature of $2M'$ vanishes (since it bounds). By Van Kampen's Theorem, $\pi_1(2M' - B) \cong \pi_1(2M')$ by inclusion (since $B$ is still collared by $S^3 \times \mathbf{R}$). Hence, we may surger away $\pi_1(2M')$ by working in the complement of $B$. The resulting closed, simply connected manifold (which we again call $M$) has a hyperbolic intersection form since it is spin with signature zero. (By a result of Wall, we may actually assume $M$ is diffeomorphic to a connected sum of $S^2 \times S^2$'s, simply by summing with enough copies of $S^2 \times S^2$.)

Now consider the open manifold $M - B$. By Casson's Embedding Theorem [1], we may represent a hyperbolic basis for $H_2(M - B)$ by Casson handles. Specifically, choose a smooth 4-ball $B'$ in $M - B$. We may then construct a family of disjoint Casson handles $\{CH_i | i = 1, \cdots, k\}$ ($k = \text{rank}\, H_2(M - B)$) which are glued onto $B'$ so that $CH_i \cup B'$ represents the $i$th basis element of $H_2(M - B)$. The attaching circles in $\partial B'$ will form $\frac{1}{2}k$ separate Hopf links (lying in disjoint 3-balls).

Freedman [3] has shown that any Casson handle is homeomorphic to an open 2-handle $D^2 \times \mathbf{R}^2$. In particular, each $CH_i$ must contain a flat topological 2-disk (corresponding to $D^2 \times 0 \subset D^2 \times \mathbf{R}^2$). Since $CH_i$ attaches to an unknot in $\partial B'$, we may attach this disk to a disk in $B'$ to obtain a flat 2-sphere $S_i$ in $CH_i \cup B'$ with $S_i \cap B'$ a smooth disk. We then have $\bigcup_{i=1}^{k} S_i$ equal to a disjoint union of $\frac{1}{2}k$ copies of $S^2 \vee S^2$, representing a hyperbolic basis of $H_2(M)$. Notice that if these were smooth, we could surger out $H_2(M)$ to obtain a smooth homotopy 4-sphere.

Our next objective is to associate to each $CH_i$ a topologically slice link $L_i$ such that if $L_i$ were smoothly slice we could make $S_i$ smoothly embedded. Let $T_i^5$ denote the first 5 stages of $CH_i$. Freedman's Reimbedding Theorems [3], [7] show that any 5-stage tower contains a Casson handle, so we may assume that $S_i$ lies in $T_i^5 \cup B'$. Consider the core disks $\{c_l\}$ of the 6th stage of $CH_i$. These are immersed disks with interiors disjoint from $T_i^5$. They can be turned into embedded disks $\{c_l'\}$ at the expense of hitting $S_i$ in $B'$. (Eliminate each self-intersection of $c_l$ by a finger move, pushing it off through $\partial c_l$ and introducing two intersections with the 5th stage core. Now push these off of the 5th stage into the 4th and continue, eventually pushing $2^5$ intersections off of the first stage into $B'$, so that we are left with $c_l'$ intersecting $S_i$ in $B'$, but $\text{int}\, c_l'$ disjoint from $T_i^5$.) The tower $T_i^5$ union with a closed tubular neighborhood $h_l$ of each $c_l'$ is a smooth 4-ball which we denote $B_i$. ($T_i^5$ is diffeomorphic to a boundary-sum of $S^1 \times D^3$'s, and each $h_l$ is a 2-handle which cancels one $S^1 \times D^3$.)

We have now constructed a family $\{B_i\}$ of disjoint smooth 4-balls associated to the Casson handles $CH_i$. Note that $B_i \cap S_j = \varnothing$, $j \neq i$. Consider $B_i \cap S_i$. $S_i$ intersects $T_i^5$ in a flat 2-disk. $B_i$ is made from $T_i^5$ by attaching 2-handles $h_l$. The core $c_l'$ of each $h_l$ hits $S_i$ transversely (in $B'$), so we may assume $S_i \cap h_l$ is a disjoint union of smooth disks, which are normal fibers to $c_l'$. Let $L_i$ be the link $S_i \cap \partial B_i$ in $\partial B_i = S^3$. Then $S_i \cap B_i$ is a disjoint union of flat disks in $B_i$ which topologically slice $L_i$. If $L_i$ were smoothly slice, we could replace these disks by a family of smooth disks, turning $S_i$ into a smoothly embedded sphere.

For §2, we need to identify the link $L_i$ specifically. Recall [3] that any 5-stage tower is represented by a certain ramified 5-fold Whitehead link. To get this, consider the tower to be a boundary-sum of $S^1 \times D^3$'s. Attaching 2-handles to a suitable family of framed circles turns the tower into $B^4$ in such a way that the attaching circle becomes unknotted. The attaching circle, together with the belt circles of the 2-handles, will form the desired Whitehead link. We now see that $L_i$ may be taken to be the 5-fold Whitehead link representing $T_i^5$. We need to check that the handles $h_l$ have been attached to $T_i^5$ with the correct framing. This will be the case as long as each 6th stage core disk $c_l$ had equal numbers of positive and negative self-intersections (which is easily arranged by adding small kinks). Note that $L_i$, as defined in the previous paragraph, consists of the attaching circle of $T_i^5$ together with many parallel copies of the belt circle of each $h_l$ (since $h_l$ intersects $S_i$ many times). This ramification of each belt circle will not be important, however, since $L_i$ will be slice whenever the corresponding Whitehead link is.

We now define the link $L$ required by Lemma 1.2. Consider $k$ disjoint 3-balls in $S^3$. For each $i = 1, \cdots, k$, embed the link $L_i$ in the $i$th 3-ball with reversed ambient orientation. Let $L$ denote the union of all of these links. We will show that $X \amalg M_L$ cannot be smoothly embedded in $B$ (which contains $R$), completing the proof.

Suppose that $X \amalg M_L$ is embedded in $B \subset M$. Let $B_0 \subset M$ denote the 0-handle of $M_L$; let $\{D_r\}$ denote the cores of the 2-handles. Recall that our previously constructed spheres $S_i$ and balls $B_i$ $(i = 1, \cdots, k)$ lie in $M - B$. Let $\{\gamma_i | i = 1, \cdots, k\}$ be a family of smooth disjoint arcs in $M$, disjoint from $X$ and each $S_j$, such that each $\gamma_i$ runs from $B_i$ to $B_0$ and hits $M_L$ and $\cup B_j$ only at the endpoints. A smooth 4-ball $\tilde{B}$ is formed from $B_0$ and $\cup_{i=1}^k B_i$ by taking the union with a closed tubular neighborhood of each $\gamma_i$. Each $S_i$ hits $\partial\tilde{B}$ in the link $L_i$ in $\partial B_i$, so $\cup_{i=1}^k S_i$ hits $\partial\tilde{B}$ in a link $L'$ equivalent to $L$ with reversed ambient orientation. On the other hand, the core disks $D_r$ of $M_L$ are glued onto the link $L$ in $\partial\tilde{B}$ (in a 3-ball in $\partial B_0$ disjoint from $L'$). But $L$ and $L'$ can be connected by a family $\{A_r\}$ of disjoint annuli in $\tilde{B}$ (essentially the trivial concordance $L \times I$ in $B^3 \times I$). Thus, we may replace the flat disks $(\cup S_i) \cap \tilde{B}$ by the smooth disks $A_r \cup D_r$ to obtain $\cup\tilde{S}_i$ in $M$, a smoothly embedded disjoint union of copies of $S^2 \vee S^2$, representing a hyperbolic basis of $H_2(M)$. We perform surgery on these to obtain a smooth homotopy 4-sphere $\Sigma$. By construction, $X$ is disjoint from $\cup\tilde{S}_i$ in $M$, so $X$ is embedded in $\Sigma$. But $X$ cannot embed in any homotopy 4-spheres. This contradiction completes the proof of Lemma 1.2.

## 2. More exotic $\mathbf{R}^4$'s

In this section we extend our family $\{R_n\}$ to obtain a doubly indexed family of distinct $\mathbf{R}^4$'s. We do this by exploiting orientations in the manner of [5]. In particular, we arrange for each $R_n$ ($n$ finite) to be embedded in a connected sum of $\mathbf{C}P^2$'s. We then show that the $\mathbf{R}^4$'s $R_{m,n} = R_m \natural \overline{R}_n$ $(m, n = 0, 1, 2, \cdots, \infty)$ are all distinct.

To accomplish this, we must sharpen our main lemmas. The handlebodies $M_L$ (and therefore $R_L$) are sometimes impossible to embed in any positive definite manifold such as $\sharp_k \mathbf{C}P^2$. Hence, we must restrict the class of allowable links $L$. In the proof of Lemma 1.2 we observed that the links $L$ may always be taken to be unions of ramified 5-fold Whitehead links. (For any fixed $l \geqslant 5$, $l$-fold Whitehead links will suffice, by using $l$-stage towers in the proof.) We restrict our class of links by only allowing Whitehead links whose clasps have a particular handedness. (In other words, the corresponding towers must have kinks with only one sign.) We then show by direct construction that the manifold $R_L$ of Lemma 1.1 may be assumed to embed in $\sharp_k \mathbf{C}P^2$ for some $k$. The appropriate analogues of Lemmas 1.1 and 1.2 are Lemmas 2.1 and 2.2, respectively, given below.

**Lemma 2.1.** *Let $L$ be a disjoint union of ramified 7-fold Whitehead links (with the various pieces lying in disjoint 3-balls). Suppose each of these Whitehead links corresponds to a 7-stage tower with only negative self-intersections at the top stage. Then $M_L$ embeds in $R_L$, a shaved $\mathbf{R}^4$ with a shaved embedding in $\sharp_k \mathbf{C}P^2$ for some $k$.*

*Proof.* We assume $L$ to be single ramified 7-fold Whitehead link, for the general case then follows by taking end-sums. Let $\overline{L}$ denote $L$ with reversed ambient orientation. The lemma follows from:

**Proposition 2.1.1.** *For some $k$, $\overline{L}$ is sliced by smooth disks in $B^4 \sharp (\sharp_k \mathbf{C}P^2)$ in such a way that $H_2(\sharp_k \mathbf{C}P^2)$ is carried by a compactum $K$ disjoint from the slice disks, with $K$ homeomorphic to $\sharp_k \mathbf{C}P^2$ minus the interior of a flat 4-ball.*

We obtain the lemma from this, as follows: Let $\{D_i\}$ be the given slice disks in $B^4 \sharp (\sharp_k \mathbf{C}P^2)$. Glue a 4-ball $B$ onto this manifold to obtain the closed manifold $\sharp_k \mathbf{C}P^2$. Tubular neighborhoods of the disks $D_i$ form 2-handles attached to $B$ in $\sharp_k \mathbf{C}P^2$. These will be attached with framing 0, since $B \cup (\bigcup D_i)$ is disjoint from $K$ and therefore nullhomologous. The attaching circles form the link $L$, so we have an embedding $M_L \hookrightarrow \sharp_k \mathbf{C}P^2$. (Note that the orientation of $\overline{L}$ has been reversed, since $S^3$ inherits opposite orientations as $\partial B$ and $\partial(B^4 \sharp (\sharp_k \mathbf{C}P^2))$.) Now let $K' = K$ minus an open topological collar of $\partial K$. $\sharp_k \mathbf{C}P^2 - K'$ is contractible, with end collared (topologically) by $S^3 \times \mathbf{R}$, so it is homeomorphic to $\mathbf{R}^4$ by Freedman's proper $h$-cobordism theorem [3]. Letting $R_L = \sharp_k \mathbf{C}P^2 - K'$, we have $M_L \hookrightarrow R_L \subset \sharp_k \mathbf{C}P^2$, and we may assume the inclusion is shaved.

**Proof of Proposition 2.1.1.** The link $\overline{L} \subset S^3 = \partial B^4$ is a 7-fold Whitehead link coresponding to some 7-stage tower $T^7$ with only positive kinks at the top stage. Let $l_0, \cdots, l_p$ denote the components of $\overline{L}$, with $l_0$ corresponding to the attaching circle of $T^7$. Then $l_1, \cdots, l_p$ form an unlink. Let $D_1, \cdots, D_p$ be the obvious smooth slice disks in $B^4$ bounded by $l_1, \cdots, l_p$. Removing tubular neighborhoods of $D_1, \cdots, D_p$ from $B^4$ leaves the tower $T^7$, with $l_0$ the attaching circle. Thus, it suffices to find a smooth disk in $T^7 \# (\#_k \mathbf{C}P^2)$ which is bounded by $l_0$ and disjoint from a suitable compactum $K$.

Each core disk at the top stage of $T^7$ has only positive kinks (self-intersections), by hypothesis. We eliminate these by "blowing up $\mathbf{C}P^2$'s", a procedure described carefully in [5] (Proof of Fact 3.4). Basically, at each kink of a core disk $c_j$ we connected sum the ambient space with $\mathbf{C}P^2$, so that the two sheets of $c_j$ are diverted through copies of $\pm\mathbf{C}P^1$ and no longer intersect. We now have $T^7 \# (\#_k \mathbf{C}P^2)$ with embedded disks $\{c_j'\}$ replacing the original 7th stage core disks $\{c_j\}$. Note that this "blowing up" procedure has not changed the homology class of $c_j$ (in homology rel the first 6 stages). This is because we have killed $(+)$ kinks with $+\mathbf{C}P^2$'s, so that the two sheets of $c_j'$ running over each $\mathbf{C}P^2$ represent opposite generators $\pm[\mathbf{C}P^1] \in H_2(\mathbf{C}P^2)$.

Now let $T^6$ denote the tower consisting of the first 6 stages of $T^7$, shrunken away from $\partial T^7$ everywhere except near the attaching circle $l_0$. Let $h_j$ be a closed tubular neighborhood of $c_j'$, for each $j$. Then $B = T^6 \cup (\bigcup h_j)$ is a smooth 4-ball in $T^7 \# (\#_k \mathbf{C}P^2)$. Each $h_j$ is a 2-handle attached to $T^6$, representing the same homology class as $c_j$ in $H_2(T^7 \# (\#_k \mathbf{C}P^2), T^6)$. Thus, the $h_j$'s are attached to $T^6$ with the "standard" framings, so $l_0$ is unknotted in $\partial B$.

Note that $H_2(T^7 \# (\#_k \mathbf{C}P^2))$ is carried by a compact submanifold $\tilde{K}$ diffeomorphic to $\#_k \mathbf{C}P^2$ minus the interior of a smooth 4-ball. In particular, let $\tilde{K}$ equal the punctured $\mathbf{C}P^2$'s we have just added, together with thickened arcs joining them together. We easily arrange $\tilde{K} \cap T^6 = \varnothing$, although $\tilde{K}$ will necessarily hit the handles $h_j$. $\tilde{K}$ is our prototype for the compactum $K$.

We will construct a particular topological disk in $T^6$, bounded by $l_0$. This disk will be a band-sum of "doubled" disks. First, we consider a model case. Let $k$ be a kinky handle with one kink; let $l$ denote its attaching circle. Glue a 2-handle $h$ to $k$ to obtain a 4-ball with $l$ unknotted in its boundary. Since $k$ is diffeomorphic to $S^1 \times D^3$, we may view it as a small collar attached to the attaching circle $l'$ of $h$. Thus, $l$ is essentially the double of $l'$. The obvious slice disk $D$ for $l$ is the double of the core disk $D'$ of $h$. That is, $D$ consists of two parallel copies of $D'$, connected by a small band with a 360° twist. (This mirrors the construction of $l$ as the double of $l'$, by a twisted band-sum of two copies of $l'$.) For further details of this construction, see Casson [1].

We now return to the tower $T^6$. Consider this to be a 1-stage tower (kinky handle) $T^1$ with 5-stage towers $\{T_r^5\}$ glued on top. From Freedman [3] we know that each $T_r^5$ contains a flat topological 2-disk $D_r'$ bounded by the attaching circle of $T_r^5$. If we thicken each $D_r'$ to obtain a topological 2-handle, we may apply the reasoning of the previous paragraph. We obtain a flat topological slice disk $D$ for $l_0$ in $T^6$, with $D$ the band-sum of the doubles of each of the disks $D_r'$.

Now consider $D$ inside the ball $B = T^6 \cup (\bigcup h_j)$. By a result of Gordon, the double of any slice disk in $B^4$ for the unknot is an unknotted disk. (See Casson [1, Lecture 2, Lemma 5.1].) Since $l_0$ is the band-sum of the doubles of the unknots $\partial D_r'$ (pushed into $\partial B$), the slice disk $D$ is unknotted. Thus, there is a topological isotopy of $B$ sending $D$ onto a smooth disk $D_0$ with the same boundary circle $l_0$. By Casson [1, Lecture 2, Lemma 0], we may assume the isotopy fixes $\partial B$ pointwise. Extend the isotopy trivially to all of $T^7 \# (\#_k \mathbf{C}P^2)$. Recall that $D$ lies in $T^6$ which is disjoint from the compactum $\tilde{K}$. Hence, the image $K$ of $\tilde{K}$ under the isotopy will be disjoint from $D_0$. $K$ is the required compactum of Proposition 2.1.1, and the slicing of $\overline{L}$ is given by $D_0$, together with the disks $D_1, \cdots, D_p$ constructed at the beginning of the proof.

**Lemma 2.2.** *Let $R$ be a shaved $\mathbf{R}^4$. Then there is a link $L$ as in Lemma* 2.1 (*i.e. a union of ramified 7-fold Whitehead links*) *such that the corresponding towers have no positive self-intersections, and such that the disjoint union $X \amalg M_L$ cannot be smoothly embedded in $R$.*

**Addendum 2.2.1.** Suppose that $R'$ is another shaved $\mathbf{R}^4$, with a shaved embedding into a closed, simply connected, negative definite manifold $M'$. Then the manifold $X \amalg M_L$ constructed in the proof of Lemma 2.2 cannot be embedded in $R \natural R'$.

**Proof of Lemma 2.2.** We merely need to sharpen the proof of Lemma 1.2. As before, we put $R \subset B \subset M$ with $M$ a homotopy $\#_l S^2 \times S^2$, and represent $H_2(M)$ by Casson handles disjoint from $B$. At this point, we eliminate all of the negative kinks in the first seven stages of the Casson handles, by blowing up $\overline{\mathbf{C}P^2}$'s as in the proof of Lemma 2.1 (and throwing away obsolete kinky handles). This does not affect the homology classes of the Casson handles, so they now represent a basis for $H_2(M)$ inside the manifold $M \# (\#_r \overline{\mathbf{C}P^2})$. We continue the proof as before, using 7-stage towers $T_i^7$ in place of the $T_i^5$. The resulting links $L_i$ will be 7-fold Whitehead links corresponding to the towers $T_i^7$, which have no negative kinks. Since the final link $L$ is the union of the $L_i$'s with reversed orientation, it will have the form required by Lemma 2.2.

To complete the proof, we suppose $X \amalg M_L$ embeds in $R$. As before, this allows us to smoothly surger out $H_2(M)$. Because of the extra $\overline{\mathbf{C}P^2}$'s, the resulting manifold will not be a homotopy sphere, but it will have a negative definite intersection form (equal to that of $\#_r \overline{\mathbf{C}P^2}$). Since $X$ does not embed in such a manifold, we still have the desired contradiction.

The addendum follows similarly. Before proving that $X \amalg M_L$ does not embed in $R$, we form the connected sum of $M \# (\#_r \overline{CP}^2)$ with $M'$ (without disturbing any of our embedded objects $B$, $T_i^7$, etc.). Now the end-sum $R \natural R'$ is embedded in $M \# (\#_r \overline{CP}^2) \# M'$, disjoint from the other objects. (This is the only place in Lemma 2.2 where we need the full strength of the hypothesis that $R \subset B \subset M$ is a shaved embedding.) If $X \amalg M_L$ embedded in $R \natural R'$, we could surger out $H_2(M)$ as before, leaving $X$ embedded in a manifold with the same (negative definite) intersection form as $(\#_r \overline{CP}^2) \# M'$.

**Theorem 2.3.** *There is a family* $\{R_{m,n} | m, n = 0, 1, 2, \cdots, \infty\}$ *of* $\mathbf{R}^4$*'s, with* $R_{m,n}$ *shaved for* $m$, $n$ *finite, such that* $R_{m,n}$ *embeds in* $R_{m',n'}$ *if and only if* $m \leqslant m'$ *and* $n \leqslant n'$. *Whenever* $m \leqslant m'$ *and* $n \leqslant n'$, $R_{m,n}$ *is an end-summand of* $R_{m',n'}$. *Otherwise, there is a compact submanifold of* $R_{m,n}$ *which cannot be embedded in* $R_{m',n'}$.

*Proof.* First we define the family $\{R_n | n = 0, 1, 2, \cdots, \infty\}$ as in Theorem 1.3, using Lemmas 2.1 and 2.2 in place of 1.1 and 1.2. Because of this improvement, each $R_n$ ($n$ finite) has a shaved embedding in $\#_k \mathbf{C}P^2$ for some (finite) $k$. (Recall that $R_1$ was defined by a shaved embedding in $\mathbf{C}P^2$. Also, whenever $R$ and $R'$ have shaved embeddings in sums of $\mathbf{C}P^2$'s $R \natural R'$ does also.)

Now define $R_{m,n}$ to be $R_m \natural \overline{R}_n$. (Note that $R_{m,0} = R_m$, since $R_0 = \mathbf{R}^4$.) To prove the nonembedding property, suppose that $m > m'$. The construction of $\{R_n\}$ gives us a compact manifold of the form $X \amalg M_L$ which embeds in $R_m$, but not in $R_{m'}$. If $n'$ is finite, then $\overline{R}_{n'}$ has a shaved embedding in $\#_k \overline{\mathbf{C}P}^2$, so Addendum 2.2.1 implies that $X \amalg M_L$ does not embed in $R_{m',n'}$. Since $X \amalg M_L$ is compact, any embedding of it in $R_{m',\infty}$ would give an embedding in $R_{m',n'}$ for some finite $n'$. Thus, $X \amalg M_L$ embeds in $R_{m,n}$ but not $R_{m',n'}$ for arbitrary $n$, $n' = 0, 1, 2, \cdots, \infty$. (To prove the case $m \leqslant m'$, $n > n'$, simply reverse orientation.)

**Remarks.** $\overline{R}_{m,n} = R_{n,m}$. $R_{m,n}$ has an orientation-reversing self-diffeomorphism if and only if $m = n$. It embeds in $\#_\infty \mathbf{C}P^2$ if and only if $n = 0$. For $m$ finite, $R_{m,0}$ embeds (shaved) in a finite sum of $\mathbf{C}P^2$'s.

We may arrange the construction so that $R_{m,n}$ is never shaved for $m$ or $n$ infinite. More generally, $R_{\infty,n}$ will not embed in any smooth 4-manifold so that the image is contained in a flat topological 4-ball. We arrange this as follows: Let $\mathscr{L}_-$ denote the set of all ramified 7-fold Whitehead links whose corresponding towers have no positive kinks. At each step of the construction of $\{R_n\}$ we obtain a link $L$ which is a union of elements of $\mathscr{L}_-$. We may add more elements of $\mathscr{L}_-$ disjointly onto $L$ at each step without affecting the argument. Since $\mathscr{L}_-$ is countable, this allows us to force $R_\infty$ to contain the disjoint union of $X$ with an infinite family of handlebodies $M_L$, where each $L \in \mathcal{L}_-$ is represented infinitely many times. (Actually, a much smaller "cofinal" family will suffice. It seems a good conjecture that such a cofinal family will be forced on us by the construction of $\{R_n\}$.) Now if $R_\infty$ (or $R_{\infty,n}$) embedded in a flat ball in any 4-manifold, Lemma 2.2 would give an immediate contradiction. (The full strength of the hypothesis that $R$ be shaved is not needed except in the addendum.)

Freedman and Taylor [4] first used this argument to show that their universal $\mathbf{R}^4$, $R_U$, does not embed in a flat 4-ball. They used the class $\mathcal{L}$ of all topologically slice links in place of $\mathcal{L}_-$.

A natural question is the following: Can (or must) $R_{\infty,\infty}$ equal $R_U$? This would follow, for example, if $M_L$ ($L$ an arbitrary 7-fold Whitehead link) could always be embedded in any $R$ in which all families $\coprod_{i=1}^{\infty} M_{L_i}$ embed, each $L_i$ or its mirror image in $\mathcal{L}_-$.

## 3. Uncountably many $\mathbf{R}^4$'s

Clifford Taubes [10] has recently proven a generalization of Donaldson's Theorem which applies to open 4-manifolds with "periodic" ends. This theorem was inspired by an observation of Freedman, that such a result would yield a family of distinct $\mathbf{R}^4$'s parametrized by $\mathbf{R}$. We expand this construction to obtain a family parametrized by $\mathbf{R}^2$. More precisely, we prove:

**Theorem 3.1.** *There is a family* $\{R_{s,t} | 0 < s, t < \infty\}$ *of (shaved) exotic* $\mathbf{R}^4$*'s such that* $R_{s,t}$ *embeds (preserving orientation) in* $R_{s',t'}$ *if and only if* $s \leqslant s'$ *and* $t \leqslant t'$.

We also obtain: If $s > s'$ or $t > t'$ there is a compact submanifold of $R_{s,t}$ which does not embed in $R_{s',t'}$. No two elements of $\{R_{s,t}\}$ have the same end. (We say that $R$ and $R'$ have the same end if some neighborhood of the end of $R$ maps to a neighborhood of the end of $R'$ by an orientation- and end-preserving diffeomorphism.)

We will state Taubes' Theorem in a form convenient for our purposes. Let $M$ be a smooth, oriented, open 4-manifold with one end.

**Definition.** $M$ will be called *end-periodic* if there exists (1) a smooth unoriented manifold $Y$ homeomorphic (but not necessarily diffeomorphic) to $S^3 \times S^1 \#(\#_n CP^2)$ for some finite $n$, and (2) a neighborhood of the end of $M$ which is diffeomorphic to a neighborhood of one end of $\tilde{Y}$, where $\tilde{Y}$ denotes the universal cover of $Y$.

Note that $H_3(Y) \cong \mathbf{Z}$ is carried by an open subset $U$, homeomorphic to $S^3 \times \mathbf{R}$, disjoint from the $CP^2$'s. We may find a smooth, closed 3-manifold

$N$ in $U$ which separates the ends of $U$. Let $W$ denote the manifold $Y$ cut open along $N$, with $\partial W = \partial_+ W \amalg \partial_- W \approx N \amalg \overline{N}$. Then the end of $M$ has a neighborhood with closure diffeomorphic to an infinite stack of copies of $W$ (formed from $W \times \{0, 1, 2, \cdots\}$ by identifying $\partial_+(W \times \{n\})$ with $\partial_-(W \times \{n+1\})$ for each $n$). An $M$ which is end-periodic by our definition is easily seen to be "admissible" in the sense of Taubes.

**Theorem** (*Taubes*). *Let $M$ be a smooth, simply connected, end-periodic 4-manifold. Suppose the intersection pairing on $H_2(M)$ is definite. Then this pairing is standard, i.e., diagonalizable over* $\mathbf{Z}$.

In particular, let $E_8$ be the even, negative definite form of rank 8, and let $Q_n$ be the standard, negative definite form of rank $n$ (possibly infinite). Then $E_8 \oplus Q_n$ is not realized by an end-periodic $M$ as above. (This pairing is not diagonalizable, since no nontrivial element of $E_8$ can be written as a linear combination of elements of square -1 in $E_8 \oplus Q_n$.)

**Proof of Theorem 3.1.** First, we construct a 1-parameter family $\{R_s | 0 < s < \infty\}$ of distinct $\mathbf{R}^4$'s. This argument is essentially due to Freedman.

In [5], a certain open 4-manifold is constructed; we shall denote it by $M_0$. This manifold is smooth and simply connected, with end collared topologically by $S^3 \times \mathbf{R}$. The intersection pairing of $M_0$ is $E_8 \oplus Q_1$. The exotic $\mathbf{R}^4$ $R_\Gamma \subset \mathbf{C}P^2$ is constructed so that a neighborhood $U$ of its end is (orientation- and end-preserving) diffeomorphic to a neighborhood of the end of $M_0$. (This is how $R_\Gamma$ is shown to be exotic.)

Let $h$: $\mathbf{R}^4 \to R_\Gamma$ be a homeomorphism. By Quinn [8], we may assume that $h$ is smooth near the positive $x_1$-axis. Let $B_r$ denote the open ball of radius $r$ about 0 in $\mathbf{R}^4$. Choose $r_0$ large enough that $h(B_{r_0}) \cup U = R_\Gamma$. (Alternatively, we may choose $r_0$ so that $X \subset h(B_{r_0})$, $X$ as in §1.) Let $R_s = h(B_{r_0+s})$ (with the smooth structure induced by $R_\Gamma$). Then $\{R_s | 0 < s < \infty\}$ is the desired family of exotic $\mathbf{R}^4$'s. Note that $R_s \subset R_{s'}$ if $s \leqslant s'$. We will prove the converse.

Suppose $R_s$ embeds (preserving orientation) in $R_{s'}$ with $s > s'$. Since $R_{s'}$ embeds in $R_s$ with compact closure, we have an embedding $i$: $R_s \hookrightarrow R_s$ with compact closure. There is a neighborhood $V$ of the end of $R_s$ which is disjoint from $i(R_s)$. We may assume that $V$ is homeomorphic to $S^3 \times \mathbf{R}$. Let $W$ denote $R_s$ minus $i(R_s - V)$, i.e., the region between $V$ and $i(V)$ (inclusive). Then $V$ and $i(V)$ are neighborhoods of the two ends of $W$, and we may identify these neighborhoods via $i$ to obtain a closed, smooth manifold $Y$. (Note that the outward-pointing end of $V$ is mapped to the end of $i(V)$ pointing into $W$, as required for this.) This manifold $Y$ is homeomorphic to $S^3 \times S^1$. (Proof: Let $S$ be the topological 3-sphere in $V$ corresponding to $S^3 \times 0$ in $S^3 \times \mathbf{R} \approx V$. The compact region $A$ between $S$ and $i(S)$ is homeomorphic to $S^3 \times I$ by the Annulus Conjecture (Quinn [8]), and $W$ is formed by identifying the two boundary components of $A$.)

We may assume $V$ lies in $U$, so there is an embedding $j\colon V \hookrightarrow M_0$, sending $V$ "near" the end of $M_0$. Let $M_1$ denote $M_0$ minus the noncompact component of $M_0 - j(V)$. Form the manifold $M$ by gluing half of the universal cover of $Y$ onto the end of $M_1$. That is, form $M$ from $M_1 \cup (W \times \{0,1,2,\cdots\})$ by identifying $j(V)$ with $i(V) \times \{0\}$, and $V \times \{n\}$ with $i(V) \times \{n+1\}$ for $n = 0,1,2,\cdots$. Since $V$ and $W$ are homeomorphic to $S^3 \times \mathbf{R}$, $M$ has the same homotopy type as $M_0$. In particular, $M$ is simply connected and has intersection form $E_8 \oplus Q_1$. But $M$ is clearly end-periodic, contradicting Taubes' Theorem.

**Remark.** We have actually shown the following: If $V$ is any open subset of $R_s$ homeomorphic to $S^3 \times \mathbf{R}$, and $V$ surrounds $R_{s'}$ (i.e., $R_{s'}$ lies in the compact component of $R_s - V$) then $V$ has no orientation-preserving embedding in $R_{s'}$. (There is no embedding of $V$ into any $\mathbf{R}^4$ which turns $V$ inside out; otherwise we could construct a closed manifold with intersection form $E_8 \oplus Q_1$.) In particular, any compact submanifold of $R_s$ which contains $V$ cannot embed in $R_{s'}$. This also implies that no two elements of $\{R_s\}$ can have the same end.

Now let $R_{s,t}$ denote the end-sum $R_s \natural \overline{R}_t$. Then $\{R_{s,t} | 0 < s, t < \infty\}$ is the desired two-parameter family of Theorem 3.1. If $s \leqslant s'$ and $t \leqslant t'$ then $R_s \subset R_{s'}$ and $\overline{R}_t \subset \overline{R}_{t'}$. These embeddings are shaved for $s < s'$ and $t < t'$ (see §1), since the homeomorphism $h\colon \mathbf{R}^4 \to R_\Gamma$ defining $R_s$ and $R_t$ was taken to be smooth near the positive $x_1$-axis. In particular, the end-sum $R_{s,t}$ embeds in $R_{s',t'}$.

Conversely, suppose that $R_{s,t}$ embeds in $R_{s',t'}$ with $s > s'$. Since $\overline{R}_{t'} \subset \overline{R}_\Gamma \subset \overline{CP}^2$ we have shaved embeddings $R_{s'} \subset R_s$ and $\overline{R}_{t'} \subset \overline{CP}^2$. Thus, $R_{s',t'}$ embeds with compact closure in $R_s \# \overline{CP}^2$. We now have $R_s \subset R_{s,t} \hookrightarrow R_{s',t'} \hookrightarrow R_s \# \overline{CP}^2$, so there is an embedding $i\colon R_s \hookrightarrow R_s \# \overline{CP}^2$ such that $i(R_s)$ has compact closure. We now repeat the argument for the one-parameter case: Let $V$ be a neighborhood of the end of $R_s \# \overline{CP}^2$, homeomorphic to $S^3 \times \mathbf{R}$, and disjoint from $i(R_s)$ and $\overline{CP}^2$. Let $W$ be the region between $V$ and $i(V)$, inclusive, and let $Y$ be the manifold formed from $W$ by gluing the ends together via $i$. This time $Y$ is homeomorphic to $S^3 \times S^1 \# \overline{CP}^2$. (Topologically, it is formed by gluing together the boundary components of $B^4 \# \overline{CP}^2$ minus the interior of a flat 4-ball.) As before, we may assume an orientation-preserving embedding $j\colon V \to M_0$. Thus, we may again construct $M$ by replacing the end of $M_0$ with half of the universal cover of $Y$. This time, the intersection form of $M$ is $E_8 \oplus Q_\infty$. But $M$ is still simply connected and end-periodic, contradicting the theorem of Taubes.

**Remarks.** Given $s, t, s', t'$ with $s > s'$, the above argument actually gives a neighborhood $U$ (homeomorphic to $S^3 \times \mathbf{R}$) of the end of $R_{s,t}$ with the following property: If $V$ (homeomorphic to $S^3 \times \mathbf{R}$) is an open subset of $U$ carrying $H_3(U)$ then $V$ has no orientation-preserving embedding in $R_{s',t'}$. (Construct $W$ in $R_{s,t} \# \overline{\mathbf{C}P}^2$; modify $M_0$ to get $M_1$ with the same end as $R_{s,t}$.) In particular, no compact submanifold of $R_{s,t}$ which contains such a $V$ embeds (preserving orientation) in $R_{s',t'}$. The same discussion applies whenever $t > t'$ ($s, s'$ arbitrary) by reversing orientations. Thus, no two elements of $\{R_{s,t}\}$ can have the same end.

We may obtain uncountably many diffeomorphism types of smooth structures on $S^3 \times \mathbf{R}$ by considering neighborhoods homeomorphic to $S^3 \times \mathbf{R}$ near the ends of the $R_{s,t}$'s. Each has the following peculiar property: If $U$ is such an exotic $S^3 \times \mathbf{R}$ and $V_1$ and $V_2$ are disjoint open sets in $U$, each homeomorphic to $S^3 \times \mathbf{R}$ and carrying $H_3(U)$, then $V_1$ and $V_2$ are not diffeomorphic. (Even orientation-reversing diffeomorphisms are ruled out by letting $Y$ be nonorientable ($S^3 \tilde{\times} S^1$) and taking its double cover.) In particular, if $S$ is any flat topological 3-sphere carrying $H_3(U)$ and $\varphi\colon U \to U$ is any diffeomorphism (not necessarily orientation-preserving) then $\varphi(S) \cap S \neq \varnothing$. Thus, the topological $S^3 \times \mathbf{R}$ structure on $U$ must be extremely complicated with respect to the smooth structure.

We may ask how the 2-parameter family $\{R_{s,t}\}$ relates to the countable family of §2. In order to compare these, we must make a minor modification of the countable family so that each $R_n$ has a shaved embedding in $R_{n+1}$ (instead of embedding as an end-summand). We may then obtain a 2-parameter family $\{R'_{s,t} | 0 \leqslant s, t \leqslant \infty\}$ with $\{R_{m,n}\}$ appearing as the integer ($\cup \infty$) lattice inside, and the family of Theorem 3.1 essentially given by $1 - \varepsilon < s, t < 1$. ($R'_{s,t} = R'_s \natural \overline{R}'_t$, $R'_0 = \mathbf{R}^4$ and $R'_\infty = R_\infty$ as given in §2.) The methods of §3 show that the members of the subfamily $\{R'_{s,t} | 1 - \varepsilon < s, t \leqslant \infty\}$ are all distinct, for sufficiently small $\varepsilon > 0$. The corresponding result may not be true, however, for the entire family. In some sense, $\{R'_{s,t} | 0 \leqslant s, t \leqslant \infty\}$ is a "maximal" 2-parameter family, since the members indexed by $\infty$ cannot be embedded with compact closure in any $\mathbf{R}^4$ (see remarks following Theorem 2.3).

## Appendix: End-sums and the monoid of $\mathbf{R}^4$'s

We generalize the notion of end-sum given in §1. Let $\mathbf{Z}^+$ denote the positive integers in the discrete topology. Let $\{R_i\}$ be any family of $n$ elements of $\mathscr{R}$,

$0 \leqslant n \leqslant \infty$, indexed by the first $n$ positive integers (in particular, by $\mathbf{Z}^+$ if $n = \infty$).

**Definition.** The *end-sum* $\natural_{i=1}^n R_i$ is defined as follows: For each $i$, let $\gamma_i$: $[0, \infty) \to R_i$ be a smooth properly embedded ray. Let $\gamma$: $[0, \infty) \times \mathbf{Z}^+ \to \mathbf{R}^4$ be a smooth proper embedding of an infinite collection of rays. For each $i$, we may attach $R_i$ to $\mathbf{R}^4$ as in §1, by gluing a copy of $I \times \mathbf{R}^3$ along the rays $\gamma_i$ in $R_i$ and $\gamma\|[0, \infty) \times \{i\}$ in $\mathbf{R}^4$. Perform this gluing simultaneously for all $R_i$, and call the resulting smooth manifold $\natural_{i=1}^n R_i$.

**Lemma A.1.** *Let $\gamma, \gamma'$: $[0, \infty) \times \mathbf{Z}^+ \to R$ be two smooth proper embeddings into $R$, a smooth manifold homeomorphic to $\mathbf{R}^4$. Then $\gamma$ and $\gamma'$ are smoothly ambiently isotopic.*

This lemma, proven at the end of the section, is a consequence of the fact that circles cannot be knotted in 4 dimensions.

It follows immediately that end-sum is well defined on oriented diffeomorphism types. Furthermore, it is independent of the order of the terms. (Use Lemma A.1 and note that any permutation of $\mathbf{Z}^+$ is a homeomorphism.) $\mathbf{R}^4$ is an identity in the following sense: Suppose $m < n$ and each $R_j$ in $\{R_i\}$ with $j > m$ is diffeomorphic to $\mathbf{R}^4$. Then $\natural_{i=1}^n R_i = \natural_{i=1}^m R_i$. (We may construct an explicit diffeomorphism which shrinks away the standard $\mathbf{R}^4$'s.) Since each $R_i$ is homeomorphic to $\mathbf{R}^4$ by a map which is smooth near $\gamma_i$ (by Quinn [8]), it follows that $\natural_{i=1}^n R_i$ is homeomorphic to $\natural_{i=1}^n \mathbf{R}^4 = \mathbf{R}^4$.

The binary operation $R_1 \natural R_2$ defined in §1 is merely $\natural_{i=1}^2 R_i$. (Simply note that $\natural_{i=1}^1 R_i = R_1$.) More generally, $\natural_{i=1}^n R_i$ ($n$ finite) is equivalent to an iterated binary sum. The independence of order now implies:

**Corollary A.2.** *The set $\mathscr{R}$ under end-sum forms a commutative monoid with involution (orientation reversal), such that countable sums are always defined and independent of order.*

**Corollary A.3.** *$(\mathscr{R}, \natural)$ has no inverses.*

*Proof.* Suppose $R_1 \natural R_2 = \mathbf{R}^4$. Then

$$R_1 = R_1 \natural \left(\natural_{i=1}^\infty \mathbf{R}^4\right) = R_1 \natural (R_2 \natural R_1) \natural (R_2 \natural R_1) \natural \cdots = \mathbf{R}^4.$$

This trick was used by Mazur to prove the Schoenflies Theorem. It also has a long history in algebra. Corollary A.3 is equivalent to a result of Stallings [9], that any smooth proper embedding $\mathbf{R}^3 \hookrightarrow \mathbf{R}^4$ is standard.

We might hope to turn $\mathscr{R}$ into a group by modding out by a suitable equivalence relation. The next corollary dashes this hope.

**Corollary A.4.** *Let $G$ be any group. Then any homomorphism $\phi$: $\mathscr{R} \to G$ is trivial.*

*Proof.* Let $R \in \mathscr{R}$. Then $R \natural (\natural_{i=1}^\infty R) = \natural_{i=1}^\infty R$, so $\phi(R) \cdot \phi(\natural_{i=1}^\infty R) = \phi(\natural_{i=1}^\infty R)$ and $\phi(R) = 1$.

**Remark.** We can put one other kind of algebraic structure on $\mathscr{R}$. Define $R_1 \leqslant R_2$ if every compact, codimension zero, smooth submanifold of $R_1$ embeds in $R_2$. This may not induce a partial ordering on $\mathscr{R}$ since there could conceivably be pairs $R_1$, $R_2$ of distinct $\mathbf{R}^4$'s with $R_1 \leqslant R_2$ and $R_2 \leqslant R_1$. If we call such pairs equivalent, however, there is an induced partial ordering on the quotient space $\mathscr{R}/\sim$. It is not difficult to check that $R_1 \leqslant R_2$ and $R_3 \leqslant R_4$ imply $R_1 \natural R_3 \leqslant R_2 \natural R_4$ (and similarly for infinite sums), so $\mathscr{R}/\sim$ inherits the monoid structure from Corollary A.2, and this structure is compatible with the partial ordering. Notice that the infinite families constructed in this paper inject into $\mathscr{R}/\sim$, where they inherit the natural order type of their index sets.

**Proof of Lemma A.1.** First we push $\gamma$ and $\gamma'$ together at the integer points. Let $\{B_i\}$ be a nested sequence of flat topological balls which exhaust $R$. For each $(m, n)$ with integer coordinates in $[0, \infty) \times \mathbf{Z}^+$, let $A_{m,n}$ be an arc from $\gamma'(m, n)$ to $\gamma(m, n)$ in $R$. Let $N_{m,n}$ be a compact regular neighborhood of $A_{m,n}$ with $A_{m,n} \subset \operatorname{int} N_{m,n}$. Arrange for the neighborhoods $N_{m,n}$ to be disjoint, and assume that each $N_{m,n}$ intersects any given $B_i$ only if one of the endpoints of $A_{m,n}$ does. Since $\gamma$ and $\gamma'$ are proper, each $B_i$ will intersect only finitely many $N_{m,n}$. For each $(m, n)$ there is an ambient isotopy with support in $N_{m,n}$ which sends $\gamma'$ to a map agreeing with $\gamma$ on a neighborhood of $(m, n)$ in $[0, \infty) \times \mathbf{Z}^+$. Since each $B_i$ hits only a finite number of $N_{m,n}$ we may combine these isotopies to form an ambient isotopy sending $\gamma'$ to $\gamma''$, a proper embedding agreeing with $\gamma$ near each integer point.

Now we use a similar procedure to isotope $\gamma''$ to $\gamma$. For each $(m, n)$ consider the 1-complex $\gamma''([m, m+1] \times \{n\}) \cup \gamma([m, m+1] \times \{n\})$. This contains an embedded circle which is spanned by a self-transverse immersed disk $D_{m,n}$ (smooth except at two boundary points). We may assume $D_{m,n}$ is disjoint from $B_i$ whenever $\partial D_{m,n}$ is (since $R - B_i$ is simply connected). Thus, each $D_{m,n}$ hits only a finite number of others, and we may assume the intersections are transverse. Use finger moves to turn $\{D_{m,n}\}$ into a family $\{D'_{m,n}\}$ of disjointly embedded disks with disjoint regular neighborhoods $K_{m,n}$. Each $B_i$ will intersect only finitely many of the $K_{m,n}$. For each $(m, n)$ there is an isotopy with support in $K_{m,n}$ sending $\gamma''$ to a map agreeing with $\gamma$ on $[m, m+1] \times \{n\}$. As before, we may combine these isotopies, obtaining an ambient isotopy sending $\gamma''$ to $\gamma$.

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